The convective terms in the energy equation are identically zero, so that the majorizing equation assumes the particularly simple form

$$(\alpha^2 + 2\alpha - {}^9/_{80})\, W + {}^9/_{20} = 0 \qquad (3.6)$$

For the existence of positive solutions of Eq. (3.6) it is necessary to consider $\alpha < \alpha^* = 0.055$. The value $\alpha^* = 0.055$ determines the radius of convergence of the series for the velocity and for its first two derivatives; it is obvious that the radius can be increased, but at the expense of a refinement in the estimate for α^*.

Convergence of the expansion (3.3) can also be established through a direct verification of the identity

$$v_k(r) = \frac{1}{k!} \frac{d^k v(r, \alpha)}{d\alpha^k}\bigg|_{\alpha=0}$$

$$v(r, \alpha) = \frac{\alpha}{\exp \alpha - (\alpha + 1)} \{\exp[\alpha(1 - r^2)] - 1\}$$

Here $v(r, \alpha)$ is the exact solution of Eq. (3.2).

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ON THE EFFECT OF INJECTION ON BOUNDARY LAYER SEPARATION

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An integral inequality is obtained for the rate of fluid injection into the boundary layer of a streamlined surface. Separation takes place when this inequality is satisfied and the pressure gradient is nonnegative. In particular, separation occurs whenever the positive injection rate is constant, independently of the magnitude of that rate. The results obtained in [1] where it was shown that separation takes place at sufficiently high injection rates, are thus refined.

1. We consider the system

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2} - \frac{dp}{dx}, \qquad \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \qquad (1.1)$$

in the domain D_a $\{0 < x < a,\ 0 < y < \infty\}$ with the conditions

$$u\mid_{y=0} = 0, \qquad v\mid_{y=0} = v_0(x), \qquad u\mid_{x=0} = u_0(y) \tag{1.2}$$

$u(x, y) \to U(x)$ for $y \to \infty$, uniformly with respect to x,

$$U^2(x) + 2p(x) = \text{const.}$$

We assume that $u_0(y) > 0$ for $y > 0$, $u_0(0) = 0$, $u_0'(0) > 0$, $u_0(y) \to U(0)$ for $y \to \infty$; dp/dx and $v_0(x)$ are continuously differentiable on $[0, a]$; $u_0(y)$, $u_0'(y)$, $u_0''(y)$ are bounded for $0 \leqslant y < \infty$ and satisfy Hölder condition. We assume also that for small y the consistency condition

$$\nu \frac{d^2 u_0(y)}{dy^2} - \frac{dp(0)}{dx} - v_0(0)\frac{du_0(y)}{dy} = O(y^2)$$

is satisfied at the point $(0, 0)$.

It was shown in [1] that for some $a > 0$ a solution (u, v) of the problem (1.1), (1.2) exists in D_a such that $\partial u / \partial y\mid_{y=0} > 0$. Let A be the upper bound of such a values. If $A < \infty$, we say that boundary layer separation occurs and we call $x_s = A$ the separation point. If $A = \infty$, we have flow without separation.

By making the change of variables

$$x = x, \qquad \psi = \psi(x, y) \tag{1.3}$$

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}, \qquad \psi(x, 0) = -\int_0^x v_0(t)\,dt$$

in the system (1.1), we reduce the latter to the Mises form

$$u\left(\nu\frac{\partial^2 u^2}{\partial \psi^2} - 2\frac{\partial u}{\partial x}\right) = 2\frac{dp}{dx} \tag{1.4}$$

When $dp/dx \equiv 0$, we obtain the well known filtration equation

$$\frac{\partial u}{\partial x} = \frac{\nu}{2}\frac{\partial^2 u^2}{\partial \psi^2} \tag{1.5}$$

Let us introduce the following notation:

$$V(0, x) = \int_0^x v_0(t)\,dt$$

With the change of variables (1.3) the domain D_a becomes $G_a\{0 < x < a, -V(0, x) < \psi < \infty\}$, and the boundary conditions (1.2) become

$$u\mid_{\psi=-V(0,x)} = 0, \quad u\mid_{x=0} = u_*(\psi), \quad u(x, \psi) \to U(x) \quad \text{for} \quad \psi \to \infty \tag{1.6}$$

where

$$u_*\left(\int_0^y u_0(\tau)\,d\tau\right) \equiv u_0(y)$$

D e f i n i t i o n. By a generalized solution of the problem (1.5), (1.6) we mean a function $u(x, \psi)$, which is continuous, nonnegative, and bounded in G_a, satisfies the conditions (1.6), and is such that:

1) the generalized derivative $\partial u^2 / \partial \psi$ exists, is square integrable in any arbitrary finite domain, and is bounded in any arbitrary half-strip of the form $\{0 < x < a,\ \delta - V(0, x) < \psi < \infty\}$ for each δ;

2) for each function f of $C^1(G_a)$ such that $f = 0$ for $\psi = -V(0, x)$, the following inequality is satisfied for $x = a$, outside of some finite domain:

$$\iint\limits_{G_a} \left[\frac{\partial f}{\partial x} u - \frac{\nu}{2} \frac{\partial f}{\partial \psi} \frac{\partial u^2}{\partial \psi}\right] dx d\psi + \int\limits_0^\infty f(0, \psi) u_*(\psi) d\psi = 0$$

A proof was given in [2] of the existence of a generalized solution of the first boundary-value problem for Eq. (1.5) in the domain $\{0 < x < a, 0 < \psi < \infty\}$. The proof is carried out in an analogous way in the case of the domain G_a. Just as in [2], it can also be proved that where the generalized solution $u(x, \psi)$ of the problem (1.5),(1.6) is positive, $u(x, \psi)$ satisfies Eq. (1.5) in the ordinary sense.

If the generalized solution $u(x, \psi)$ of the problem (1.5),(1.6) vanishes inside G_a, separation of the boundary layer occurs at the point $x_s < a$. In fact, if in D_a a solution of the problem (1.1),(1.2) exists such that $u > 0$ in D_a and $\partial u / \partial y|_{y=0} > 0$, then a positive solution $u(x, \psi)$ of the problem (1.5),(1.6) exists in G_a, which satisfies Eq. (1.5) in the ordinary sense.

2. Let us consider a Cauchy problem for the filtration equation (1.5) with the initial condition

$$u|_{x=0} = u_1(\psi) \tag{2.1}$$

Definition. By a generalized solution of the problem (1.5),(2.1) in the strip $H_a \{0 < x < a, -\infty < \psi < \infty\}$ we mean a continuous nonnegative function $u(x, \psi)$, bounded in H_a and satisfying the condition (2.1), and such that

1) the generalized derivative $\partial u^2 / \partial \psi$ exists, is square integrable in any arbitrary finite domain, and is bounded in the strip H_a;

2) for each f of $C^1(H_a)$, $f = 0$ for $x = a$, and outside of some finite domain the equality

$$\iint\limits_{H_a} \left[\frac{\partial f}{\partial x} u - \frac{\nu}{2} \frac{\partial f}{\partial \psi} \frac{\partial u^2}{\partial \psi}\right] dx d\psi + \int\limits_{-\infty}^\infty f(0, \psi) u_1(\psi) d\psi = 0$$

is satisfied.

The existence of a generalized solution $u(x, \psi)$ of the problem (1.5),(2.1) was proved in [2] under the assumption that $u_1(\psi)$ is contunuous, $0 \leqslant u_1(\psi) \leqslant M_0$, and the function $u_1^2(\psi)$ satisfies Lipschitz condition. Some properties of the function $u(x, \psi)$ were established in [3]. In particular, in Lemma 1 of [3] it was shown that if $u_1(\psi) = 0$ for $\psi_0 - l \leqslant \psi \leqslant \psi_0 + l$, $l > 0$, then $u(x, \psi_0) = 0$ for $0 \leqslant x \leqslant x_0$, where $x_0 > 0$ is determined from the relation

$$\frac{l^2}{6\nu x_0} = M_0 = \sup u_1(\psi) \tag{2.2}$$

In the sequel, $u_1(\psi)$ has the special form

$$u_1(\psi) = \begin{cases} u_*(\psi), & \psi \geqslant 0 \\ 0, & \psi < 0 \end{cases} \tag{2.3}$$

The function $u_1(\psi)$ defined in this way is continuous, bounded, and $u_1^2(\psi)$ satisfies Lipschitz condition.

Let $u(x, \psi)$ be a generalized solution of the Cauchy problem (1.5),(2.3). By Theorem 21 of [2], for each $x_0 > 0$ there exists $\psi_*(x_0) \leqslant 0$ such that $u(x_0, \psi) = 0$ for $\psi \leqslant \psi_*(x_0)$. As was proved in [2, 3], the curve $\psi_*(x)$ divides the halfplane $\{x \geqslant 0\}$ into two parts: one to the left of the curve, where $u(x, \psi) = 0$ and the other to the right of it, where $u(x, \psi) > 0$. In [3] it was also proved that $\psi_*(x)$ is a continuous nonincreasing function. We now establish yet another property of the curve $\psi_*(x)$.

Lemma. Let $M_0 = \sup u_1(\psi)$. Then $\psi_*(x) \geqslant -\sqrt{6\nu M_0 x}$.

Proof. For each $l > 0$, we have $u_1(\psi) = 0$ for $-2l \leqslant \psi \leqslant 0$. Therefore, according to Lemma 1 of [3], $u(x, -l) = 0$ for $x \leqslant x_0$, where x_0 is determined from Eq. (2.2). Hence $u(x, \psi) = 0$ for $\psi \leqslant -\sqrt{6\nu M_0 x}$ and, consequently,

$$\psi_*(x) \geqslant -\sqrt{6\nu M_0 x} \tag{2.4}$$

3. Let $u(x, \psi)$ be a generalized solution of the Cauchy problem (1.5), (2.3), and let u_b be a generalized solution of the first boundary-value problem (1.5), (1.6). Since $u(0, \psi) = u_b(0, \psi)$ for $\psi \geqslant 0$, $u(x, \psi) \geqslant 0$ for $x \geqslant 0$, and $u_{b|\psi=-V(0, x)} = 0$, it then follows from the maximum principle that $u(x, \psi) \geqslant u_b(x, \psi)$ in $G_a\{0 < x < a, -V(0, x) < \psi < \infty\}$ for all $a > 0$. Consequently, if $u(x_0, \psi_0) = 0$, where (x_0, ψ_0) belongs to G_a, it follows that $u_b(x_0, \psi_0) = 0$ and, as pointed out above, the boundary layer separation takes place.

Theorem. Let $dp/dx \geqslant 0$. If x_1 and x_2 exist such that $x_1 < x_2$ and

$$\int_{x_1}^{x_2} v_0(t)\,dt > \sqrt{6\nu M_0 (x_2 - x_1)} \tag{3.1}$$

then boundary layer separation occurs at the point $x_s < x_2$

Proof. First, we carry out the proof for the case $dp/dx \equiv 0$. We assume that for all $a > 0$ a solution of the problem (1.1), (1.2) exists in D_a such that $u(x, y) > 0$ for $y > 0$ and $\partial u/\partial y|_{y=0} > 0$. Then a positive solution $u_b(x, \psi)$ of the problem (1.5), (1.6) exists in G_a for $a = x_2$. It is obvious that $u_b(x, \psi)$ is a solution of Eq. (1.5) in the domain $H_1\{x_1 \leqslant x \leqslant x_2 = a, -V(0, x) < \psi < \infty\}$ with the boundary conditions

$$u|_{x=x_1} = u_b(x_1, \psi), \qquad u|_{\psi=-V(0, x)} = 0$$

By the maximum principle we note that $\sup_\psi u_b(x, \psi) \leqslant M_0$.

Let $u^1(x, \psi)$ denote the solution of the Cauchy problem for the Eq. (1.5) in the half-plane $x \geqslant x_1$ with the condition

$$u^1(x, \psi)|_{x=x_1} = \begin{cases} u_b(x_1, \psi), & \psi \geqslant -V(0, x_1) \\ 0, & \psi \leqslant -V(0, x_1) \end{cases} \tag{3.2}$$

To the solution $u^1(x, \psi)$ of the problem (1.5), (3.2) there corresponds a curve $\psi_*^1(x)$ issuing from the point $(x_1, -V(0, x))$. In accord with the definition of the curve $\psi_*^1(x)$ we have $u^1(x, \psi) = 0$ for $\psi \leqslant \psi_*^1(x)$. As in the lemma, we can show that

$$\psi_*^1(x) + \int_0^{x_1} v_0(t)\,dt \geqslant -\sqrt{6\nu M_0 (x - x_1)}$$

From this, using the inequality (3.1), we obtain

$$\psi_*^1(x_2) \geqslant -\int_0^{x_1} v_0(t)\,dt - \sqrt{6\nu M_0(x_2 - x_1)} > -\int_0^{x_1} v_0(t)\,dt - \int_{x_1}^{x_2} v_0(t)\,dt = -\int_0^{x_2} v_0(t)\,dt$$

This inequality implies that the intersection of the domain H_1 with the set of points (x, ψ) in which $u^1(x, \psi) = 0$ is not empty. Since $u^1(x, \psi) \geqslant u_b(x, \psi)$ in H_1, it follows that $u_b(x, \psi) = 0$ in some subdomain of the domain H_1 and, as a result, we have boundary layer separation.

Let us suppose now that $dp/dx \geqslant 0$. Consider the problem (1.4), (1.6). As shown in [1], a positive solution $u_p(x, \psi)$ of the problem (1.4), (1.6) in the domain G_a for some a is obtained as the limit of solutions $u_p^\varepsilon(x, \psi)$ of the first boundary-value problem for

Eq. (1.4) with the boundary conditions

$$u|_{\psi=-V(0,x)} = f_\varepsilon(x, \psi), \quad u|_{x=0} = g_\varepsilon(\psi), \quad u(x, \psi) \xrightarrow[\psi\to\infty]{} U(x) \qquad (3.3)$$

Here $f_\varepsilon > 0$, $g_\varepsilon > 0$; $f_\varepsilon(x, \psi) \to 0$, $g_\varepsilon(\psi) \to u_*(\psi)$ for $\varepsilon \to 0$. In addition, f_ε, g_ε are smooth functions and the consistency condition is satisfied at the point (0, 0). In the same way we obtain a positive solution $u_b(x, \psi)$ of the problem (1.5),(1.6) in G_a for some a as the limit of solutions $u_b^\varepsilon(x, \psi)$ of the problem (1.5),(3.3) as $\varepsilon \to 0$.

The function $s = u_b^\varepsilon - u_p^\varepsilon$ satisfies the linear equation

$$\frac{\partial s}{\partial x} - 2\left(\frac{\partial u_b^\varepsilon}{\partial \psi} + \frac{\partial u_p^\varepsilon}{\partial \psi}\right)\frac{\partial s}{\partial \psi} = (u_b^\varepsilon + u_p^\varepsilon)\frac{\partial^2 s}{\partial \psi^2} - \left(\frac{\partial^2 u_b^\varepsilon}{\partial \psi^2} + \frac{\partial^2 u_p^\varepsilon}{\partial \psi^2}\right)s + \frac{1}{u_p^\varepsilon}\frac{dp}{dx}$$

Since $s = 0$ on the boundary of the domain G_a, $dp / dx \geqslant 0$, $u_b^\varepsilon > 0$, $u_p^\varepsilon > 0$, and the second derivatives of the functions u_b^ε and u_p^ε are bounded with respect to ψ, it then follows from the maximum principle that $u_p^\varepsilon \leqslant u_b^\varepsilon$ in G_a. By a limiting transition we obtain the inequality $u_p(x, \psi) \leqslant u_b(x, \psi)$ in G_a From this it follows that when $dp / ax \geqslant 0$ boundary layer separation takes place in D_a for $a = x_2$ if separation takes place in D_a for $a = x_2$, when $dp / dx \equiv 0$. This completes the proof of the theorem.

Corollary. If $dp / dx \geqslant 0$ and $v_0(x) = m = \text{const} > 0$, boundary layer separation takes place in D_a for some a.

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ON SINGLE-VALUED SOLUTION OF THE FUNDAMENTAL BOUNDARY-VALUE PROBLEM IN THE THEORY OF THE THERMAL BOUNDARY LAYER

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We consider the system of thermal boundary layer equations for a two-dimensional steady forced-convective flow of an incompressible fluid. Our principal object of study being the equation for the temperature. We prove the single-valued solvability of the fundamental boundary-value problem for this equation.

The problem of the single-valued solvability of the fundamental problems of